

The cost of approximate controllability for semilinear heat equations in one space dimension

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Abstract: This note deals with the approximate controllability for the semilinear heat equation in one space dimension. Our aim is to provide an estimate of the cost of the control.

Keywords: Cost of approximate controllability, Semilinear heat equation.

1 Introduction and main result

In this paper, we apply a successful combination of three key tools which allows to get a measure of the cost of the approximate controllability for semilinear heat equation. The first tool consists to get enough information about the approximate control for the linear heat equation with a potential depending on space-time variable. Then a fixed point method is applied. The fixed point technique described here was previously used in [Z] to prove the exact controllability for semilinear wave equation in one dimension. The last tool, usually used for control problem (see [FCZ2, p.589] e.g.), consists to choose adequately the time of controllability.

Many results exist by now concerning the approximate controllability for semilinear heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ when the control acts in a non-empty subdomain $\omega \subset \Omega$, $\omega \neq \Omega$ (see [FPZ], [K] or [FCZ2] and references therein). In particular, it is proved in [FCZ2] that for any time $T > 0$, if the system

$$\begin{cases} \partial_t u - \Delta u + f(u) = h \cdot 1_\omega & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) = u_o & \text{in } \Omega , \end{cases} \quad (1.1)$$

with $f : \mathbb{R} \rightarrow \mathbb{R}$ locally lipschitz-continuous, admits at least one globally defined and bounded solution u^* , corresponding to the data $u_o^* \in L^2(\Omega)$ and $h^* \in L^\infty(\omega \times (0, T))$, and further if the function f satisfies

$$|f'(s)| \leq c(1 + |s|^p) \quad \text{a.e., with } p \leq 1 + 4/n \text{ and } c > 0 ,$$

and

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s| \ln^{3/2}(1 + |s|)} = 0 ,$$

then for any $u_o \in L^2(\Omega)$, $u_d \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control $h \in L^\infty(\omega \times (0, T))$ such that the solution of (1.1) is globally defined in $[0, T]$ and satisfies

$$\|u(\cdot, T) - u_d\|_{L^2(\Omega)} \leq \varepsilon .$$

However, in [FCZ2], no information was given about a measure of the control with respect to ε . In this paper, we provide an estimate of the control but under more restrictive hypothesis. Our result is

Theorem .- Let $\Omega = (0, 1)$ and $T > 0$. Assume $f \in C^1(\mathbb{R})$ and

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s| \sqrt{\ln(1 + |s|)}} = 0 ,$$

then, for any $(u_o, u_d) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and any $\varepsilon \in (0, 1]$, there exist a control $h_\varepsilon \in L^2(\omega \times (0, T))$ and a function $u = u(x, t) \in L^\infty(\Omega \times (0, T))$ such that

$$\|h_\varepsilon\|_{L^2(\omega \times (0, T))} \leq \exp\left(e^{C/\varepsilon}\right) ,$$

$$\|u(\cdot, T) - u_d\|_{L^2(\Omega)} \leq \varepsilon ,$$

and

$$\begin{cases} \partial_t u - \partial_{xx} u + f(u) = h_\varepsilon \cdot 1_\omega & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) = u_o & \text{in } \Omega . \end{cases} \quad (1.2)$$

Here, C is a positive constant independent on ε .

Remark .- Notice that we do not assume $f(0) = 0$. If $f(0) = 0$ (which correspond to the case $u^* = 0$), we can use the following control strategy to provide an estimate of the control when $u_o \in L^2(\Omega)$: we divide the time interval $(0, T)$ in two subintervals. During the first time interval $(0, T/2]$, we use a null control to steer the semilinear heat equation starting from u_o to zero (see [FCZ2]). In the second time interval $(T/2, T)$, we apply the above Theorem with null initial data.

The rest of this note is devoted to the proof of Theorem.

2 Proof of Theorem

We proceed in three steps.

Step 1 .- Preliminary on the cost of the approximate controllability for the linear heat equation with a potential. We first recall some results from [P] concerning the cost of the approximate controllability for the heat equation with a potential $a = a(x, t) \in L^\infty(\Omega \times (0, T))$. We denote $\|a\|_\infty = \|a\|_{L^\infty(\Omega \times (0, T))}$. In the sequel, $c_1 > 1$ and $c_2 > 1$ are two constants only depending on Ω and ω . Let $T' \in (0, T]$ called time of controllability of the linear system. We introduce the operator \mathbf{C} given by

$$\mathbf{C} : \vartheta \in L^2(\omega \times (0, T')) \longrightarrow w(\cdot, 0) \in L^2(\Omega) ,$$

where $w \in C([0, T']; H_0^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$ is the solution of

$$\begin{cases} -\partial_t w - \Delta w + aw = E\vartheta \cdot 1_\omega & \text{in } \Omega \times (0, T') , \\ w = 0 & \text{on } \partial\Omega \times (0, T') , \\ w(\cdot, T') = 0 & \text{in } \Omega , \end{cases}$$

with $a \in L^\infty(\Omega \times (0, T))$ and $E = \exp\left(c_2\left(1 + T'\|a\|_\infty\left(1 + e^{c_2 T'\|a\|_\infty^2}\right) + \|a\|_\infty^{2/3}\right)\right)$. We define $\mathcal{F} = \text{Im } \mathbf{C}$ the space of exact controllability initial data with the following norm :

$$\|w_o\|_{\mathcal{F}} = \inf \left\{ \|\vartheta\|_{L^2(\omega \times (0, T'))} \mid \mathbf{C}\vartheta = w_o \right\} . \quad (2.1)$$

Denote \mathbf{C}^* the adjoint of \mathbf{C} . It has been proved (see [P]) that the operator $\mathbf{B} = \mathbf{C}\mathbf{C}^*$ is non-negative, compact and self-adjoint on $L^2(\Omega)$ which allows us to associate the Hilbert basis with eigenfunctions ξ_n of \mathbf{B} and eigenvalues $\mu_n > 0$ where μ_n is non-increasing and tends to zero. Furthermore, let the sets $S_n = \{m > 0 / \alpha_{n+1} < \mu_m \leq \alpha_n\}$ where

$$\alpha_n = e^{\mu_1 + e} e^{-e^n} , \quad (2.2)$$

for all $n > 0$, then each function $\phi \in L^2(\Omega)$ can be represented in the form $\phi = \sum_{n>0} \phi_n$ where $\phi_n = \sum_{m \in S_n} (\phi, \xi_m) \xi_m$. Finally, let $N > 0$ and $z \in H_0^1(\Omega)$, then we can write, in $L^2(\Omega)$:

$$z = \sum_{n \leq N} z_n + \sum_{n > N} z_n \text{ with } z_n = \sum_{m \in S_n} (z, \xi_m) \xi_m ,$$

with the properties

$$\begin{aligned} \left\| \sum_{n \leq N} z_n \right\|_{\mathcal{F}} &\leq c_3 \frac{1}{\sqrt{\alpha_{N+1}}} \|z\|_{L^2(\Omega)} , \\ \left\| \sum_{n \leq N} z_n - z \right\|_{L^2(\Omega)} &\leq c_3 \frac{D}{\ln\left(2 + \frac{1}{\sqrt{\alpha_{N+1}}}\right)} \|z\|_{H_0^1(\Omega)} , \end{aligned} \quad (2.3)$$

for some constant $c_3 > 0$ independent on N , z , T' and a and where $D = c_1 \left(T' e^{c_1 T'} \|a\|_\infty^2 + \frac{1}{T'} \right) > 1$ (see [P]). Here, $\sum_{n \leq N} z_n \in \mathcal{F}$ and precisely

$$\begin{aligned} \sum_{n \leq N} z_n &= \sum_{n \leq N} \sum_{m \in S_n} (z, \xi_m) \xi_m \\ &= \mathbf{C} \left(\sum_{n \leq N} \sum_{m \in S_n} (z, \xi_m) \frac{1}{\mu_m} \mathbf{C}^* \xi_m \right) . \end{aligned}$$

On another hand, let $\chi \cdot 1|_\omega$ be the null-control function which steers to zero at time T' the solution of the heat equation with potential $a(x, T' - t)$ and initial data $\pi_o \in L^2(\Omega)$. It is known (see [FCZ1]) that

$$\|\chi\|_{L^2(\omega \times (0, T'))} \leq G \|\pi_o\|_{L^2(\Omega)} , \quad (2.4)$$

where $G = \exp\left(c_0 \left(1 + \frac{1}{T'} + T' \|a\|_\infty + \|a\|_\infty^{2/3}\right)\right)$ for some constant $c_0 > 0$ only depending on Ω and ω .

Therefore, for all $T' \in (0, T]$, $a \in L^\infty(\Omega \times (0, T))$, $\pi_o \in L^2(\Omega)$, $z \in H_0^1(\Omega)$, if we choose

$$\ell(x, T' - t) = E \sum_{n \leq N} \sum_{m \in S_n} (z, \xi_m) \frac{1}{\mu_m} \mathbf{C}^* \xi_m$$

then from (2.1), (2.2), (2.3) and (2.4), the solution $v_1 \in C([0, T']; H_0^1(\Omega)) \cap W^{1,2}(0, T'; L^2(\Omega))$ of

$$\begin{cases} \partial_t v_1 - \Delta v_1 + a(x, T' - t) v_1 = (\chi + \ell) \cdot 1|_\omega & \text{in } \Omega \times (0, T') , \\ v_1 = 0 & \text{on } \partial\Omega \times (0, T') , \\ v_1(\cdot, 0) = \pi_o & \text{in } \Omega , \end{cases}$$

satisfies

$$\|v_1(\cdot, T') - z\|_{L^2(\Omega)} \leq c_4 D e^{-N} \|z\|_{H_0^1(\Omega)} , \quad (2.5)$$

and moreover

$$\|\chi + \ell\|_{L^2(\omega \times (0, T'))} \leq G \|\pi_o\|_{L^2(\Omega)} + c_4 E e^{e^N} \|z\|_{L^2(\Omega)} , \quad (2.6)$$

for any $N \geq N_o$ where $N_o > 0$ and $c_4 \geq e^{N_o}$. Clearly, the approximate-control function ℓ depends on N , z and a coming from E and the Hilbert basis (ξ_n, μ_n) .

Next, let us introduce the operator \mathbf{S} given by

$$\mathbf{S} : \lambda \in \mathbb{R} \longrightarrow v_2(\cdot, T') \in H_0^1(\Omega) ,$$

where $v_2 \in C([0, T']; H_0^1(\Omega)) \cap W^{1,2}(0, T'; L^2(\Omega))$ is the unique solution of

$$\begin{cases} \partial_t v_2 - \Delta v_2 + a(x, T' - t) v_2 = \lambda & \text{in } \Omega \times (0, T') , \\ v_2 = 0 & \text{on } \partial\Omega \times (0, T') , \\ v_2(\cdot, 0) = 0 & \text{in } \Omega , \end{cases}$$

One can easily check that

$$\|\mathbf{S}(\lambda)\|_{H_0^1(\Omega)} = \|\nabla v_2(\cdot, T')\|_{L^2(\Omega)} \leq |\lambda| \sqrt{T'} e^{c_5 T'} \|a\|_\infty^2 , \quad (2.7)$$

for some constant $c_5 > 0$ only depending on Ω and ω .

Consequently, for all $T' \in (0, T]$, $a \in L^\infty(\Omega \times (0, T))$, $\pi_o \in L^2(\Omega)$, $z_d \in H_0^1(\Omega)$, if we choose $z = z_d - \mathbf{S}(\lambda)$ the solution $v_3 = v_1 + v_2 \in C([0, T']; H_0^1(\Omega)) \cap W^{1,2}(0, T'; L^2(\Omega))$ of

$$\begin{cases} \partial_t v_3 - \Delta v_3 + a(x, T' - t) v_3 = \lambda + (\chi + \ell) \cdot 1|_\omega & \text{in } \Omega \times (0, T') , \\ v_3 = 0 & \text{on } \partial\Omega \times (0, T') , \\ v_3(\cdot, 0) = \pi_o & \text{in } \Omega , \end{cases}$$

satisfies, taking into account (2.5), (2.6) and (2.7),

$$\|v_3(\cdot, T') - z_d\|_{L^2(\Omega)} \leq c_4 D e^{-N} \left(\|z_d\|_{H_0^1(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T'} \|a\|_\infty^2 \right) ,$$

and

$$\|\chi + \ell\|_{L^2(\omega \times (0, T'))} \leq G \|\pi_o\|_{L^2(\Omega)} + c_4 E e^{e^N} \left(\|z_d\|_{L^2(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T'} \|a\|_\infty^2 \right) .$$

Finally, let $q \in L^\infty(\Omega \times (0, T))$. Now, we conclude with the construction of a solution v of the heat equation with a potential and a second member and with a control acting on the interval $(T - T', T)$. Precisely, we divide the time interval $(0, T)$ in two subintervals. During the first time interval $(0, T - T')$, we let the system

$$\begin{cases} \partial_t v - \Delta v + qv = \lambda & \text{in } \Omega \times (0, T - T') , \\ v = 0 & \text{on } \partial\Omega \times (0, T - T') , \\ v(\cdot, 0) = u_o & \text{in } \Omega , \end{cases}$$

to evolve freely without control. In the second time interval $(T - T', T)$, we choose $a(\cdot, t) = q(\cdot, T - t)$, $\pi_o = v(\cdot, T - T')$ and the control function such that

$$\begin{cases} \partial_t v - \Delta v + qv = \lambda + [(\chi + \ell)(x, T' - T + t)] \cdot 1|_{\omega \times (T - T', T)} & \text{in } \Omega \times (0, T) , \\ v = 0 & \text{on } \partial\Omega \times (0, T) , \\ v(\cdot, 0) = u_o & \text{in } \Omega , \end{cases}$$

satisfies

$$\|v(\cdot, T) - z_d\|_{L^2(\Omega)} \leq c_4 D e^{-N} \left(\|z_d\|_{H_0^1(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T'} \|q\|_\infty^2 \right) ,$$

and moreover

$$\|\chi + \ell\|_{L^2(\omega \times (0, T'))} \leq G \|v(\cdot, T - T')\|_{L^2(\Omega)} + c_4 E e^{e^N} \left(\|z_d\|_{L^2(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T'} \|q\|_\infty^2 \right) ,$$

for any $N \geq N_o$ where $N_o > 0$ and $c_4 \geq e^{N_o}$. Notice that one can easily check that

$$\|v(\cdot, T - T')\|_{L^2(\Omega)} \leq e^{c_6 T} \|q\|_\infty^2 \left(\|u_o\|_{L^2(\Omega)} + c_6 |\lambda| \sqrt{T} \right) ,$$

for some constant $c_6 > 0$ only depending on Ω and ω .

Choosing

$$N \leq \ln \left(c_4 D e^{\frac{1+\varepsilon}{\varepsilon}} \left(1 + \|z_d\|_{H_0^1(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T'} \|q\|_\infty^2 \right) \right) < N + 1$$

then one has

$$\|v(\cdot, T) - z_d\|_{L^2(\Omega)} \leq \varepsilon ,$$

and moreover,

$$\begin{aligned} \|\chi + \ell\|_{L^2(\omega \times (0, T'))} &\leq G e^{c_6 T \|q\|_\infty^2} \left(\|u_o\|_{L^2(\Omega)} + c_6 |\lambda| \sqrt{T} \right) \\ &\quad + c_4 E \exp \left(c_4 D e^{\frac{1+\varepsilon}{\varepsilon}} \left(1 + \|z_d\|_{H_0^1(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T' \|q\|_\infty^2} \right) \right) \\ &\quad \cdot \left(\|z_d\|_{L^2(\Omega)} + |\lambda| \sqrt{T'} e^{c_5 T' \|q\|_\infty^2} \right) . \end{aligned}$$

Step 2 .- Introduction of g and choice of T' . We begin to fix $\varepsilon \in (0, 1]$ and $(u_o, u_d) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Next, we introduce

$$g(s) = \begin{cases} \frac{f(s) - f(0)}{s} & \text{for } s \neq 0 \\ f'(0) & \text{at } s = 0 \end{cases}$$

which satisfies, from our hypothesis on f , the following assertion

$$\forall \delta > 0 \quad \exists C_\delta > 0 \quad \forall s \in \mathbb{R} \quad |g(s)| \leq C_\delta + \delta \sqrt{\ln(1 + |s|)} ,$$

and consequently, for any $u \in L^\infty(\Omega \times (0, T))$, $g(u) \in L^\infty(\Omega \times (0, T))$ and one has

$$\forall \delta > 0 \quad \exists C_\delta > 0 \quad \|g(u)\|_\infty \leq C_\delta + \delta \sqrt{\ln(1 + \|u\|_\infty)} .$$

Hence, we easily deduce that

$$\forall \delta > 0 \quad \exists C_\delta > 0 \quad \exp \left(\frac{1}{\delta} \|g(u)\|_\infty^2 \right) \leq C_\delta + \|u\|_\infty . \quad (2.8)$$

Now, we take $T' \in (0, T]$ depending on ε and $\|g(u)\|_\infty$ as follows

$$T' = \begin{cases} T & \text{if } \varepsilon \|g(u)\|_\infty^2 \leq 1 \\ \frac{T}{\varepsilon \|g(u)\|_\infty^2} & \text{if } \varepsilon \|g(u)\|_\infty^2 > 1 \end{cases} \quad (2.9)$$

Step 3 .- The fixed point method thanks to the homotopy invariance of the Leray-Schauder degree. In order to prove Theorem, we will apply the homotopical version of the Leray-Schauder fixed point theorem.

Theorem (Leray-Schauder) .- Let \mathcal{E} be a Banach space and $\mathbf{H} : \mathcal{E} \times [0, 1] \rightarrow \mathcal{E}$ be a compact continuous mapping such that $\mathbf{H}(u, 0) = 0$ for every $u \in \mathcal{E}$. If there exists a constant K such that $\|u\|_{\mathcal{E}} < K$ for every pair $(u, \sigma) \in \mathcal{E} \times [0, 1]$ satisfying $u = \mathbf{H}(u, \sigma)$, then the mapping $\mathbf{H}(\cdot, 1) : \mathcal{E} \rightarrow \mathcal{E}$ has a fixed point.

We introduce the following mapping \mathbf{H}

$$\mathbf{H} : (u, \sigma) \in L^\infty(\Omega \times (0, T)) \times [0, 1] \longrightarrow \sigma y \in L^\infty(\Omega \times (0, T)) ,$$

where $y \in C([0, T]; H_0^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$ is the solution of

$$\begin{cases} \partial_t y - \Delta y + \sigma g(u) y = -\sigma f(0) + h \cdot 1|_\omega & \text{in } \Omega \times (0, T) , \\ y = 0 & \text{on } \partial\Omega \times (0, T) , \\ y(\cdot, 0) = u_o & \text{in } \Omega , \end{cases}$$

when the control function h depends on (u, σ) as follows: from $q = \sigma g(u) \in L^\infty(\Omega \times (0, T))$, we take $a(\cdot, t) = q(\cdot, T - t)$ and generate the eigencouple (ξ_n, μ_n) , next we choose the control function

$$h(x, T - t) = \begin{cases} 0 & \text{for } T' \leq t < T \\ \chi(x, T' - t) + E \sum_{n \leq N} \sum_{m \in S_n} (u_d - \mathbf{S}(-\sigma f(0)), \xi_m) \frac{1}{\mu_m} \mathbf{C}^* \xi_m & \text{for } 0 < t < T' \end{cases}$$

where $N \geq N_o$ is such that $N \leq \ln \left(c_4 D e^{\frac{1+\varepsilon}{\varepsilon}} \left(1 + \|u_d\|_{H_0^1(\Omega)} + \sigma |f(0)| \sqrt{T'} e^{c_5 T' \|\sigma g(u)\|_\infty^2} \right) \right) < N + 1$, then the unique solution y satisfies

$$\|y(\cdot, T) - u_d\|_{L^2(\Omega)} \leq \varepsilon,$$

and moreover, one has

$$\begin{aligned} \|h\|_{L^2(\omega \times (0, T))} &\leq G e^{c_6 T \|\sigma g(u)\|_\infty^2} \left(\|u_o\|_{L^2(\Omega)} + c_6 \sigma |f(0)| \sqrt{T'} \right) \\ &\quad + c_4 E \exp \left(c_4 D e^{\frac{1+\varepsilon}{\varepsilon}} \left(1 + \|u_d\|_{H_0^1(\Omega)} + \sigma |f(0)| \sqrt{T'} e^{c_5 T' \|\sigma g(u)\|_\infty^2} \right) \right) \\ &\quad \cdot \left(\|u_d\|_{L^2(\Omega)} + \sigma |f(0)| \sqrt{T'} e^{c_5 T' \|\sigma g(u)\|_\infty^2} \right), \end{aligned} \quad (2.10)$$

with

$$\begin{cases} G = \exp \left(c_0 \left(1 + \frac{1}{T'} + T' \|\sigma g(u)\|_\infty + \|\sigma g(u)\|_\infty^{2/3} \right) \right), \\ D = c_1 \left(T' e^{c_1 T' \|\sigma g(u)\|_\infty^2} + \frac{1}{T'} \right) > 1, \\ E = \exp \left(c_2 \left(1 + T' \|\sigma g(u)\|_\infty e^{c_2 T' \|\sigma g(u)\|_\infty^2} + \|\sigma g(u)\|_\infty^{2/3} \right) \right). \end{cases} \quad (2.11)$$

Clearly, the control function h depends on ε , u_o , u_d and (u, σ) coming from E and the eigencouple (ξ_n, μ_n) .

From now, we use the letter c to denote a positive constant only depending on Ω and ω , whose value can change from line to line. From (2.10) and (2.11), the control function is bounded as follows: for any $\varepsilon \in (0, 1]$, $(u_o, u_d) \in L^2(\Omega) \times H_0^1(\Omega)$ and $T' \in (0, T]$, $T > 0$,

$$\begin{aligned} \|h\|_{L^2(\omega \times (0, T))} &\leq \left(\|u_o\|_{L^2(\Omega)} + \sigma |f(0)| \sqrt{T'} \right) \exp \left(c \left(1 + T \|\sigma g(u)\|_\infty^2 + \frac{1}{T'} + T' \|\sigma g(u)\|_\infty + \|\sigma g(u)\|_\infty^{2/3} \right) \right) \\ &\quad + \left(\|u_d\|_{L^2(\Omega)} + \sigma |f(0)| \sqrt{T'} \right) \exp \left(c \left(1 + T' \|\sigma g(u)\|_\infty^2 + T' \|\sigma g(u)\|_\infty e^{c T' \|\sigma g(u)\|_\infty^2} + \|\sigma g(u)\|_\infty^{2/3} \right) \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(T' e^{c T' \|\sigma g(u)\|_\infty^2} + \frac{1}{T'} \right) \left(1 + \|u_d\|_{H_0^1(\Omega)} + \sigma |f(0)| \sqrt{T'} e^{c T' \|\sigma g(u)\|_\infty^2} \right) \right) \end{aligned} \quad (2.12)$$

and therefore

$$\begin{aligned} \|h\|_{L^2(\omega \times (0, T))} &\leq \left(\|u_o\|_{L^2(\Omega)} + \sigma |f(0)| \sqrt{T'} \right) \exp \left(c \left(1 + T \|\sigma g(u)\|_\infty^2 + \frac{1}{T'} + T' \|\sigma g(u)\|_\infty + \|\sigma g(u)\|_\infty^{2/3} \right) \right) \\ &\quad + \left(\|u_d\|_{L^2(\Omega)} + \sigma |f(0)| \sqrt{T'} \right) \exp \left(c \left(1 + T' \|\sigma g(u)\|_\infty^2 + \sqrt{T'} e^{c T' \|\sigma g(u)\|_\infty^2} + \|\sigma g(u)\|_\infty^{2/3} \right) \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(1 + T'^2 + T' \left(\|u_d\|_{H_0^1(\Omega)} + |\sigma f(0)|^2 \right) \right) e^{c T' \|\sigma g(u)\|_\infty^2} \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon T'} \left(1 + \|u_d\|_{H_0^1(\Omega)} + |\sigma f(0)|^2 \right) \right). \end{aligned} \quad (2.13)$$

The continuity and compactness property of \mathbf{H} comes from the following embedding

$$W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \subset L^\infty(\Omega \times (0, T))$$

which is compact in one dimension of space. It remains to prove that

$$\|u\|_\infty < K,$$

for every pair $(u, \sigma) \in L^\infty(\Omega \times (0, T)) \times [0, 1]$ satisfying $u = \mathbf{H}(u, \sigma)$.

The solution u of the nonlinear system $\mathbf{H}(u, \sigma) = u$ is also solution of the linear system

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + q(x, t) \psi = b(x, t) & \text{in } \Omega \times (0, T) , \\ \psi = 0 & \text{on } \partial\Omega \times (0, T) , \\ \psi(\cdot, 0) = \sigma u_o & \text{in } \Omega , \end{cases}$$

by substituting $q = \sigma^2 g(u)$ and $b = \sigma(-\sigma f(0) + h \cdot 1|_\omega)$. But such solution ψ satisfies, in one space dimension, the following inequality

$$\|\psi\|_\infty^2 \leq c e^{cT \|q\|_\infty^2} \left(\|\sigma u_o\|_{H_0^1(\Omega)}^2 + \|b\|_{L^2(\Omega \times (0, T))}^2 \right) .$$

Consequently, the later inequality and (2.13) imply that

$$\begin{aligned} \|u\|_\infty^2 &\leq c e^{cT \|g(u)\|_\infty^2} \left(\|u_o\|_{H_0^1(\Omega)}^2 + |f(0)T|^2 + \|h\|_{L^2(\Omega \times (0, T))}^2 \right) \\ &\leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \\ &\quad \cdot \exp \left(c \left(1 + T \|g(u)\|_\infty^2 + \frac{1}{T'} + \sqrt{T'} e^{cT' \|g(u)\|_\infty^2} + T' \|g(u)\|_\infty + \|g(u)\|_\infty^{2/3} \right) \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(1 + T'^2 + T' \left(\|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) e^{cT' \|g(u)\|_\infty^2} \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon T'} \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) . \end{aligned}$$

First, if $\varepsilon \|g(u)\|_\infty^2 \leq 1$, then we easily get an uniform bound for u in $L^\infty(\Omega \times (0, T))$,

$$\begin{aligned} \|u\|_\infty^2 &\leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \\ &\quad \cdot \exp \left(c \left(1 + \frac{T}{\varepsilon} + \frac{1}{T'} + \sqrt{T'} e^{cT/\varepsilon} + \frac{T}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon^{1/3}} \right) \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(1 + T^2 + T \left(\|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) e^{cT/\varepsilon} \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon T} \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) \end{aligned}$$

which gives

$$\|u\|_\infty^2 \leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \exp \left(C_T \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) e^{cT/\varepsilon} \right) ,$$

where $C_T > 0$ is a constant only dependent on T , Ω and ω .

Now if $\varepsilon \|g(u)\|_\infty^2 > 1$ then by the choice of T' given by (2.9), we have

$$\begin{aligned} \|u\|_\infty^2 &\leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \\ &\quad \cdot \exp \left(c \left(1 + T \|g(u)\|_\infty^2 + \frac{\varepsilon \|g(u)\|_\infty^2}{T} + \sqrt{T'} e^{cT/\varepsilon} + T \|g(u)\|_\infty + \|g(u)\|_\infty^{2/3} \right) \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(1 + T^2 + T \left(\|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) e^{cT/\varepsilon} \right) \\ &\quad \cdot \exp \left(c \frac{\|g(u)\|_\infty^2}{T} \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) \end{aligned}$$

which gives

$$\begin{aligned} \|u\|_\infty^2 &\leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \\ &\quad \cdot \exp \left(c \left(1 + T + \frac{1}{T'} \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) \|g(u)\|_\infty^2 \right) \\ &\quad \cdot \exp \left(\frac{c}{\varepsilon} \left(1 + T^2 + T \left(\|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) \right) e^{cT/\varepsilon} \right) \end{aligned}$$

and finally, using (2.8), there exists a constant $C' > 0$ only depending on $\left(\|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right)$, T , Ω and ω such that

$$\|u\|_\infty^2 \leq \left(\|u_o\|_{H_0^1(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)T|^2 \right) \exp \left(C_T \left(1 + \|u_d\|_{H_0^1(\Omega)} + |f(0)|^2 \right) e^{cT/\varepsilon} \right) (C' + \|u\|_\infty) ,$$

where $C_T > 0$ is a constant only dependent on T , Ω and ω .

We conclude that any solution $(u, \sigma) \in L^\infty(\Omega \times (0, T)) \times [0, 1]$ of $u = \mathbf{H}(u, \sigma)$ satisfies the following estimate: there is a constant $C > 0$ independent of (u, σ) such that for any $\varepsilon \in (0, 1]$,

$$\|u\|_\infty^2 \leq \exp\left(e^{C/\varepsilon}\right) ,$$

which allows us to get to the existence of a fixed point for $\mathbf{H}(\cdot, 1)$. Furthermore, by (2.13), the control is then bounded as follows: for any $\varepsilon \in (0, 1]$,

$$\|h\|_{L^2(\omega \times (0, T))} \leq \exp\left(e^{C/\varepsilon}\right) .$$

This completes the proof.

Remark .- Notice that the measure of the cost of the control of the semilinear heat equation (1.2) can be improved and become of order e^{C/ε^2} by adding the following more restrictive hypothesis $f(0) = 0$ and

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s| \sqrt{|\ln \ln(1 + |s|)|}} = 0 .$$

Indeed, the minimization of the second member of (2.12) with respect to the quantity $\|g(u)\|_\infty$ suggests us our choice (2.9) of the time of controllability T' . But the minimization of the second member of (2.12) when $f(0) = 0$ with respect to $\varepsilon \in (0, 1]$, suggests to take $T' = \varepsilon T$ in order to get an estimate of the cost of order e^{C/ε^2} .

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